

# On entanglement in neutrino mixing and oscillations

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**Abstract.** We report on recent results about entanglement in the context of particle mixing and oscillations. We study in detail single-particle entanglement arising in two-flavor neutrino mixing. The analysis is performed first in the context of Quantum Mechanics, and then for the case of Quantum Field Theory.

## 1. Introduction

Entanglement has been widely investigated in a number of physical systems, ranging from condensed matter to atomic physics, and quantum optics [1]. Also in the context of particle physics, the role of entanglement has been considered, see for instance Refs. [2].

In this paper, we consider the entanglement associated to neutrino mixing and oscillations. A detailed study of such a topic has been performed recently in the context of quantum mechanics [3, 4]. Here we review the main results of these studies in the simplest case of two flavors. We show that these results suggest a simple extension of the analysis to the relativistic domain, thus providing the physical basis and the mathematical tools for the quantification of entanglement in quantum field theory.

The phenomenon of particle mixing, associated with a mismatch between flavor and mass of the particle, appears in several instances: quarks, neutrinos, and the neutral  $K$ -meson system [5, 6]. Particle mixing is at the basis of important effects as neutrino oscillations and  $CP$  violation [7]. Flavor mixing for the case of three generations is described by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) in the lepton instance [8, 9].

In the following we consider only the simplest case of two flavors. In such a case, the PMNS matrix reduces to the  $2 \times 2$  rotation Pontecorvo matrix  $\mathbf{U}(\theta)$ ,

$$\mathbf{U}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad (1)$$

which connects the neutrino states with definite flavor with those with definite masses:

$$|\underline{\nu}^{(f)}\rangle = \mathbf{U}(\theta) |\underline{\nu}^{(m)}\rangle \quad (2)$$

where  $|\underline{\nu}^{(f)}\rangle = (|\nu_e\rangle, |\nu_\mu\rangle)^T$  and  $|\underline{\nu}^{(m)}\rangle = (|\nu_1\rangle, |\nu_2\rangle)^T$ .

From Eq. (2), we see that each flavor state is given by a superposition of mass eigenstates, i.e.  $|\nu_\alpha\rangle = U_{\alpha 1}|\nu_1\rangle + U_{\alpha 2}|\nu_2\rangle$ . Let us recall that both  $\{|\nu_\alpha\rangle\}$  and  $\{|\nu_i\rangle\}$  are orthonormal, i.e.  $\langle\nu_\alpha|\nu_\beta\rangle = \delta_{\alpha,\beta}$  and  $\langle\nu_i|\nu_j\rangle = \delta_{i,j}$ .

We now establish the following correspondence with two-qubit states:

$$|\nu_1\rangle \equiv |1\rangle_1|0\rangle_2 \equiv |10\rangle, \quad |\nu_2\rangle \equiv |0\rangle_1|1\rangle_2 \equiv |01\rangle, \quad (3)$$

where  $|\rangle_i$  denotes states in the Hilbert space for neutrinos with mass  $m_i$ . Thus, the occupation number allows to interpret the flavor states as constituted by entangled superpositions of the mass eigenstates. Quantum entanglement emerges as a direct consequence of the superposition principle. It is important to remark that the Fock space associated with the neutrino mass eigenstates is physically well defined. In fact, at least in principle, the mass eigenstates can be produced or detected in experiments performing extremely precise kinematical measurements [10]. In this framework, as discussed in Ref. [3], the quantum mechanical state (2) is entangled in the field modes, although being a single-particle state.

Mode entanglement defined for single-photon states of the radiation field or associated with systems of identical particles has been discussed in Ref. [11]. The concept of mode entanglement in single-particle states has been widely discussed and is by now well established [11, 12]. Successful experimental realizations using single-photon states have been reported as well [13]. Moreover, remarkably, the nonlocality of single-photon states has been experimentally demonstrated [14], verifying a theoretical prediction [15]. Furthermore, the existing schemes to probe nonlocality in single-particle states have been generalized to include massive particles of arbitrary type [16].

In the dynamical regime, flavor mixing (and neutrino mass differences) generates the phenomenon of neutrino oscillations. The mass eigenstates  $|\nu_j\rangle$  have definite masses  $m_j$  and definite energies  $\omega_j$ . Their propagation can be described by plane wave solutions of the form  $|\nu_j(t)\rangle = e^{-i\omega_j t}|\nu_j\rangle$ . The time evolution of the flavor neutrino states Eq.(2) is given by:

$$|\underline{\nu}^{(f)}(t)\rangle = \tilde{\mathbf{U}}(t)|\underline{\nu}^{(f)}\rangle, \quad \tilde{\mathbf{U}}(t) \equiv \mathbf{U}(\theta) \mathbf{U}_0(t) \mathbf{U}(\theta)^{-1}, \quad (4)$$

where  $|\underline{\nu}^{(f)}\rangle$  are the flavor states at  $t = 0$ ,  $\mathbf{U}_0(t) = \text{diag}(e^{-i\omega_1 t}, e^{-i\omega_2 t})$ , and  $\tilde{\mathbf{U}}(t = 0) = \mathbb{I}$ .

At time  $t$ , the probability associated with the transition  $\nu_\alpha \rightarrow \nu_\beta$  is

$$P_{\nu_\alpha \rightarrow \nu_\beta}(t) = |\langle\nu_\beta|\nu_\alpha(t)\rangle|^2 = |\tilde{\mathbf{U}}_{\alpha\beta}(t)|^2, \quad (5)$$

where  $\alpha, \beta = e, \mu$ . The explicit form for the transition probabilities in the two flavor case is:

$$P_{\nu_e \rightarrow \nu_e}(t) = 1 - \sin^2(2\theta) \sin^2\left(\frac{\omega_2 - \omega_1}{2}t\right), \quad (6)$$

$$P_{\nu_e \rightarrow \nu_\mu}(t) = \sin^2(2\theta) \sin^2\left(\frac{\omega_2 - \omega_1}{2}t\right). \quad (7)$$

Flavor neutrino states are well defined in the context of Quantum Field Theory (QFT), where they are obtained as eigenstates of the flavor neutrino charges [17, 18]. In the relativistic limit, the exact QFT flavor states reduce to the usual Pontecorvo flavor states Eq.(2): flavor modes are thus legitimate and physically well-defined individual entities and mode entanglement can be defined and studied in analogy with the static case of Ref.[3]. We can thus establish the following correspondence with two-qubit states:

$$|\nu_e\rangle \equiv |1\rangle_e|0\rangle_\mu, \quad |\nu_\mu\rangle \equiv |0\rangle_e|1\rangle_\mu. \quad (8)$$

States  $|0\rangle_\alpha$  and  $|1\rangle_\alpha$  correspond, respectively, to the absence and the presence of a neutrino in mode  $\alpha$ . Entanglement is thus established among flavor modes, in a single-particle setting. Eq. (4) can then be recast as

$$|\nu_\alpha(t)\rangle = \tilde{\mathbf{U}}_{\alpha e}(t)|1\rangle_e|0\rangle_\mu + \tilde{\mathbf{U}}_{\alpha\mu}(t)|0\rangle_e|1\rangle_\mu, \quad (9)$$

with the normalization condition  $\sum_\beta |\tilde{\mathbf{U}}_{\alpha\beta}(t)|^2 = 1$  ( $\alpha, \beta = e, \mu$ ). The time-evolved states  $|\underline{\nu}^{(f)}(t)\rangle$  are entangled superpositions of the two flavor eigenstates with time-dependent coefficients. Thus, flavor oscillations can be related to bipartite (flavor) entanglement of single-particle states [4].

## 2. Entanglement in neutrino mixing – Quantum Mechanics

As discussed in the Introduction, the flavor neutrino state at a given time, say  $|\nu_e(t)\rangle$  for definiteness, can be regarded as an entangled state either in terms of the mass eigenstates or in terms of the flavor eigenstates (at a fixed time). In the first instance, which was studied in detail for the multipartite case in Ref.[3], we have a static entanglement, in the sense that the result of the entanglement measures on the state  $|\nu_e(t)\rangle$  do not depend on time. In the second case, considered for the general three flavor case in Ref.[4], the entanglement varies with time as it is related to the oscillations of flavor(s).

In this Section, we discuss these two forms of entanglement in the simple case of two flavors, by means of entropy measures first, and then using the characterization of entanglement in terms of quantum uncertainties. The second approach turns out to be suitable for the generalization of our discussion to the case of mixing among relativistic quantum fields – see next Section.

### 2.1. Neutrino entanglement via linear entropy

In terms of the mass eigenstates, the electron neutrino state at time  $t$  reads:

$$|\nu_e(t)\rangle = e^{-i\omega_1 t} \cos\theta |\nu_1\rangle + e^{-i\omega_2 t} \sin\theta |\nu_2\rangle, \quad (10)$$

where  $|\nu_i\rangle$  are interpreted as the qubits, see Eq.(3).

Following the usual procedure, we construct the density operator  $\rho^{(\alpha)} = |\nu_\alpha(t)\rangle\langle\nu_\alpha(t)|$  corresponding to the pure state  $|\nu_\alpha(t)\rangle$ . Then we consider the density matrix  $\rho_i^{(\alpha)} = \text{Tr}_j[\rho^{(\alpha)}]$  reduced with respect to the index  $j$ . For the specific case of the state Eq.(10), we have  $\rho^{(e)} = |\nu_e(t)\rangle\langle\nu_e(t)|$  and

$$\rho_1^{(e)} = \text{Tr}_2[\rho^{(e)}] = \cos^2\theta |1\rangle_1\langle 1| + \sin^2\theta |0\rangle_1\langle 0| \quad (11)$$

$$\rho_2^{(e)} = \text{Tr}_1[\rho^{(e)}] = \cos^2\theta |0\rangle_2\langle 0| + \sin^2\theta |1\rangle_2\langle 1| \quad (12)$$

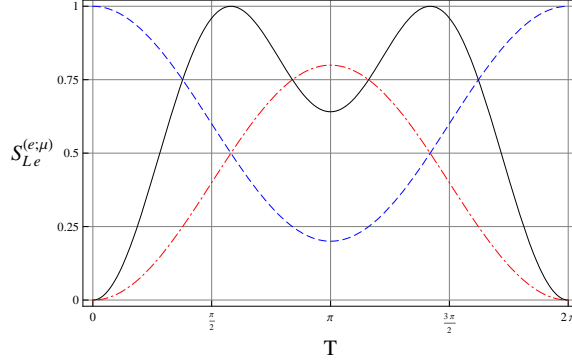
It is then easy to calculate the corresponding linear entropies, which turn out to be equal:

$$S_L^{(1;2)}(\rho_e) = 2 \left( 1 - \text{Tr}_1[(\rho_1^{(e)})^2] \right) = \sin^2(2\theta), \quad (13)$$

$$S_L^{(2;1)}(\rho_e) = 2 \left( 1 - \text{Tr}_2[(\rho_2^{(e)})^2] \right) = \sin^2(2\theta), \quad (14)$$

Similar results are found for the muon neutrino state. Note that the above results are particular cases of the more general ones obtained for the three flavor neutrino states in Ref.[3], where it was found that such states can be classified as generalized W states. In the present (two-flavor) case, the form of the entangled state is simply that of a Bell state.

Eqs.(13)-(14) express the fact that flavor neutrino states at any time can be regarded as entangled superpositions of the mass qubits  $|\nu_i\rangle$ , where the entanglement is a function of the mixing angle only.



**Figure 1.** (Color online) Linear entropy  $S_L^{(e;\mu)}(\rho_e)$  (full) as a function of the scaled time  $T = \frac{2Et}{\Delta m_{12}^2}$ . The mixing angle  $\theta$  is fixed at the experimental value  $\sin^2 \theta = 0.314$ . The transition probabilities  $P_{\nu_e \rightarrow \nu_e}$  (dashed) and  $P_{\nu_e \rightarrow \nu_\mu}$  (dot-dashed) are reported as well for comparison.

Let us now turn to the dynamic entanglement arising in connection with flavor oscillations. To this aim, we rewrite the electron neutrino state  $|\nu_e(t)\rangle$  as

$$|\nu_e(t)\rangle = \tilde{\mathbf{U}}_{ee}(t)|\nu_e\rangle + \tilde{\mathbf{U}}_{e\mu}(t)|\nu_\mu\rangle, \quad (15)$$

where  $|\nu_e\rangle, |\nu_\mu\rangle$  are the flavor neutrino states at time  $t = 0$  and are now taken as the relevant qubits (see Eq.(8)). By proceeding in a similar way as done for the static case, we arrive at the following expression for the linear entropies associated to the above state:

$$\begin{aligned} S_L^{(\mu;e)}(\rho_e) = S_L^{(e;\mu)}(\rho_e) &= 4|\tilde{\mathbf{U}}_{ee}(t)|^2 |\tilde{\mathbf{U}}_{e\mu}(t)|^2 \\ &= 4|\tilde{\mathbf{U}}_{ee}(t)|^2 (1 - |\tilde{\mathbf{U}}_{ee}(t)|^2) \end{aligned} \quad (16)$$

Eq.(16) establishes that the linear entropy of the reduced state is equal to the product of the two-flavor transition probabilities given in Eqs.(5)-(7). It is remarkable that simple expressions similar to those of Eq. (16) hold also for the three flavor case [4].

Note also that, for any reduced state  $\rho$  of a two-level system one has that  $S_L = 2[1 - \text{Tr}(\rho^2)] = 4\text{Det}\rho = 4\lambda_1(1 - \lambda_1)$ , where  $\lambda_1$  is one of the two non-negative eigenvalues of  $\rho$ , and the relation  $\lambda_1 + \lambda_2 = 1$  has been exploited. Comparing with Eq. (16), one sees that the transition probabilities coincide with the eigenvalues of the reduced state density matrix.

In Fig. 1 we show the behavior of  $S_L^{(e;\mu)}(\rho_e)$  as a function of the scaled, dimensionless time  $T = \frac{2Et}{\Delta m_{12}^2}$ . In the same figure, we also report the behavior of the transition probabilities  $P_{\nu_e \rightarrow \nu_e}$  and  $P_{\nu_e \rightarrow \nu_\mu}$ . The plots have a clear physical interpretation. At time  $T = 0$ , the entanglement is zero, the global state of the system is factorized, and the two flavors are not mixed. For  $T > 0$ , flavors start to oscillate and the entanglement is maximal at largest mixing:  $P_{\nu_e \rightarrow \nu_e} = P_{\nu_e \rightarrow \nu_\mu} = 0.5$ , and minimum at  $T = \pi$ .

## 2.2. Neutrino entanglement via uncertainties

An alternative characterization of entanglement can be given in terms of quantum uncertainties [21, 22]. In order to apply this formalism to the case of neutrino mixing and oscillations we introduce the (fermionic) annihilation operator  $\alpha_i$  for a neutrino with mass  $m_i$ , with anti-commutator  $\{\alpha_i, \alpha_j\} = \delta_{ij}$ . We then define neutrino states with definite masses as:

$$|\nu_i\rangle \equiv \alpha_i^\dagger |0\rangle_m, \quad i = 1, 2 \quad (17)$$

where  $|0\rangle_m \equiv |0\rangle_1 \otimes |0\rangle_2$  is the vacuum for the mass eigenstates.

Next we define the flavor annihilation operators by means of the following relations:

$$\alpha_e(t) = \cos \theta \alpha_1(t) + \sin \theta \alpha_2(t) \quad (18)$$

$$\alpha_\mu(t) = -\sin \theta \alpha_2(t) + \cos \theta \alpha_1(t) \quad (19)$$

where  $\alpha_i(t) = e^{i\omega_i t} \alpha_i$ , with  $i = 1, 2$ .

The flavor states are given by:

$$|\nu_\sigma(t)\rangle \equiv \alpha_\sigma^\dagger(t) |0\rangle_m, \quad \sigma = e, \mu. \quad (20)$$

We use in the following the notation  $|\nu_\sigma\rangle \equiv |\nu_\sigma(t=0)\rangle$ .

The hamiltonian for the system is

$$\begin{aligned} H &= \omega_{ee} \alpha_e^\dagger(t) \alpha_e(t) + \omega_{\mu\mu} \alpha_\mu^\dagger(t) \alpha_\mu(t) + \omega_{e\mu} \left( \alpha_e^\dagger(t) \alpha_\mu(t) + \alpha_\mu^\dagger(t) \alpha_e(t) \right) \\ &= \omega_1 \alpha_1^\dagger \alpha_1 + \omega_2 \alpha_2^\dagger \alpha_2 \end{aligned} \quad (21)$$

where we used the relations  $\omega_{ee} = \omega_1 \cos^2 \theta + \omega_2 \sin^2 \theta$ ,  $\omega_{\mu\mu} = \omega_1 \sin^2 \theta + \omega_2 \cos^2 \theta$ ,  $\omega_{e\mu} = (\omega_2 - \omega_1) \sin \theta \cos \theta$ .

In the mass basis, we can introduce the  $SU(2)$  operators (the superscript  $m$  stands for mass):

$$J_+^m = \alpha_1^\dagger \alpha_2, \quad J_-^m = \alpha_2^\dagger \alpha_1, \quad (22)$$

$$J_3^m = \frac{1}{2} (\alpha_1^\dagger \alpha_1 - \alpha_2^\dagger \alpha_2) \quad (23)$$

$$\mathcal{C} = \frac{1}{2} (\alpha_1^\dagger \alpha_1 + \alpha_2^\dagger \alpha_2) \quad (24)$$

We also have

$$N_1 = \mathcal{C} + J_3^m \quad (25)$$

$$N_2 = \mathcal{C} - J_3^m \quad (26)$$

$$H = \omega_1 N_1 + \omega_2 N_2 \quad (27)$$

The static entanglement of the electron neutrino state  $|\nu_e(t)\rangle$  defined in Eq.(20), is characterized, in the present formalism, by the variances associated with the numbers  $N_i$ , relative to the mass qubits. Thus we have:

$$\begin{aligned} \Delta N_i(\nu_e) &\equiv \langle \nu_e(t) | N_i^2 | \nu_e(t) \rangle - \langle \nu_e(t) | N_i | \nu_e(t) \rangle^2 \\ &= \frac{1}{4} \sin^2(2\theta), \quad i = 1, 2. \end{aligned} \quad (28)$$

This result differs by a factor 4 from that obtained by means of the linear entropy, Eqs.(13)-(14).

In order to discuss the dynamical entanglement of the state  $|\nu_e(t)\rangle$ , we need to introduce flavor oscillations, which can be seen both in terms of overlaps of states at different times:

$$P_{\nu_e \rightarrow \nu_e}(t) = |\langle \nu_e | \nu_e(t) \rangle|^2 \quad (29)$$

$$P_{\nu_e \rightarrow \nu_\mu}(t) = |\langle \nu_\mu | \nu_e(t) \rangle|^2 \quad (30)$$

with  $P_{\nu_e \rightarrow \nu_e}(t) + P_{\nu_e \rightarrow \nu_\mu}(t) = 1$ , or equivalently in terms of expectation values of number operators at time  $t$ :

$$P_{\nu_e \rightarrow \nu_e}(t) = \langle \nu_e | N_e(t) | \nu_e \rangle \quad (31)$$

$$P_{\nu_e \rightarrow \nu_\mu}(t) = \langle \nu_e | N_\mu(t) | \nu_e \rangle \quad (32)$$

$$N_\sigma(t) = \alpha_\sigma^\dagger(t) \alpha_\sigma(t) \quad \sigma = e, \mu \quad (33)$$

The explicit expressions of the transition probabilities are given in Eqs.(6),(7).

In the flavor basis, we can again introduce  $SU(2)$  operators (at time  $t$ ):

$$J_+^f(t) = \alpha_e^\dagger(t) \alpha_\mu(t) \quad , \quad J_-^f(t) = \alpha_\mu^\dagger(t) \alpha_e(t) \quad (34)$$

$$J_3^f(t) = \frac{1}{2} \left( \alpha_e^\dagger(t) \alpha_e(t) - \alpha_\mu^\dagger(t) \alpha_\mu(t) \right) \quad (35)$$

$$\mathcal{C} = \frac{1}{2} \left( \alpha_e^\dagger(t) \alpha_e(t) + \alpha_\mu^\dagger(t) \alpha_\mu(t) \right) = \frac{1}{2} \left( \alpha_1^\dagger \alpha_1 + \alpha_2^\dagger \alpha_2 \right) \quad (36)$$

where the superscript  $f$  stands for flavor. The flavor number operators are:

$$N_e(t) = \mathcal{C} + J_3^f(t) \quad (37)$$

$$N_\mu(t) = \mathcal{C} - J_3^f(t) \quad (38)$$

Note that the Hamiltonian can be written as:

$$H = \omega_{ee} N_e(t) + \omega_{\mu\mu} N_\mu(t) + \omega_{e\mu} (J_+^f(t) + J_-^f(t)) \quad (39)$$

Flavor entanglement is given by the variances of the above flavor numbers. Consider first the the variances of the  $SU(2)$  operators in the flavor basis, which give:

$$\Delta J_1(\nu_e)(t) = \frac{1}{4} \left[ 1 - \sin^2(4\theta) \sin^4 \left( \frac{\omega_2 - \omega_1}{2} t \right) \right] \quad (40)$$

$$\Delta J_2(\nu_e)(t) = \frac{1}{4} - \sin^2(2\theta) \sin^2[(\omega_2 - \omega_1)t] \quad (41)$$

$$\Delta J_3(\nu_e)(t) = \sin^2(2\theta) \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right) \left[ 1 - \sin^2(2\theta) \sin^2 \left( \frac{\omega_2 - \omega_1}{2} t \right) \right] \quad (42)$$

$$\Delta \mathcal{C}(\nu_e) = 0 \quad (43)$$

When we restrict to a given flavor (e.g. electron neutrinos), we find that the entanglement is given by

$$\Delta N_e(\nu_e)(t) \equiv \langle \nu_e(t) | N_e^2(t) | \nu_e(t) \rangle - \langle \nu_e(t) | N_e(t) | \nu_e(t) \rangle^2 \quad (44)$$

$$= P_{\nu_e \rightarrow \nu_e}(t) P_{\nu_e \rightarrow \nu_\mu}(t) \quad (45)$$

with the same result for  $\Delta N_\mu(\nu_e)(t)$ . The above result coincides (again up to a factor 4) with the one obtained in Eq.(16) by means of the linear entropy.

Analogous results are easily obtained for the state  $|\nu_\mu(t)\rangle$ .

### 3. Neutrino mixing in Quantum Field Theory

We now look for an extension of the above discussion to a relativistic context. To this aim, we need to consider neutrino mixing at level of fields [17]. For two flavors, the mixing transformations are

$$\nu_e(x) = \cos \theta \nu_1(x) + \sin \theta \nu_2(x) \quad (46)$$

$$\nu_\mu(x) = -\sin \theta \nu_1(x) + \cos \theta \nu_2(x), \quad (47)$$

where  $\nu_e(x)$  and  $\nu_\mu(x)$  are the Dirac neutrino fields with definite flavors. Here,  $\nu_1(x)$  and  $\nu_2(x)$  are the free neutrino fields with definite masses  $m_1$  and  $m_2$ , respectively. For the purpose of discussing the quantization of flavor fields and the phenomenon of neutrino oscillations, it is sufficient to consider only the free Lagrangian term for these two free fields:

$$\mathcal{L}_\nu(x) = \bar{\nu}_m(x) \left( i \not{\partial} - M_\nu^d \right) \nu_m(x), \quad (48)$$

where  $\nu_m^T = (\nu_1, \nu_2)$  and  $M_\nu^d = \text{diag}(m_1, m_2)$ .  $\mathcal{L}_\nu(x)$  is invariant under global  $U(1)$  phase transformations of the type  $\nu_m(x) = e^{i\alpha} \nu_m(x)$ . This implies the conservation of the Noether charge  $Q = \int I^0(x) d^3\mathbf{x}$  (with  $I^\mu(x) = \bar{\nu}_m(x) \gamma^\mu \nu_m(x)$ ) which is indeed the total charge of the system, i.e. the total lepton number of neutrinos.

Consider now the global  $SU(2)$  transformation [19]:

$$\nu'_m(x) = e^{i\alpha_j \cdot \tau_j} \nu_m(x) \quad j = 1, 2, 3. \quad (49)$$

with  $\alpha_j$  real constants,  $\tau_j = \sigma_j/2$  with  $\sigma_j$  being the Pauli matrices.

$\mathcal{L}_\nu$  is not invariant under the transformations (49) since  $m_1 \neq m_2$ . By use of the equations of motion, we obtain

$$\delta \mathcal{L}_\nu = i\alpha_j \bar{\nu}_m(x) \left[ \tau_j, M_\nu^d \right] \nu_m(x) = -\alpha_j \partial_\mu J_{m,j}^\mu(x), \quad (50)$$

where the currents are:

$$J_{m,j}^\mu(x) = \bar{\nu}_m(x) \gamma^\mu \tau_j \nu_m(x), \quad j = 1, 2, 3. \quad (51)$$

The related charges

$$Q_{m,j}(t) = \int d^3\mathbf{x} J_{m,j}^0(x), \quad (52)$$

satisfy the  $su(2)$  algebra:

$$[Q_{m,i}(t), Q_{m,j}(t)] = i\varepsilon_{ijk} Q_{m,k}(t). \quad (53)$$

The Casimir operator is proportional to the total (conserved) charge:  $Q_{m,0} = \frac{1}{2}Q_\nu$  and also  $Q_{m,3}$  is conserved, due to the fact that  $M_\nu^d$  is diagonal. This implies that the charges for  $\nu_1$  and  $\nu_2$  are separately conserved. The  $U(1)$  Noether charges associated with  $\nu_1$  and  $\nu_2$  can be then expressed as

$$Q_1 \equiv \frac{1}{2}Q + Q_{m,3}; \quad Q_2 \equiv \frac{1}{2}Q_\nu - Q_{m,3}. \quad (54)$$

$$Q_i = \int d^3\mathbf{x} \nu_i^\dagger(x) \nu_i(x), \quad (55)$$

with  $Q$  total (conserved) charge and  $i = 1, 2$ .

Let us now consider the Lagrangian  $\mathcal{L}_\nu(x)$  written in the flavor basis

$$\mathcal{L}_\nu(x) = \bar{\nu}_f(x) (i \not{\partial} - M_\nu) \nu_f(x), \quad (56)$$

where  $\nu_f^T = (\nu_e, \nu_\mu)$  and  $M_\nu = \begin{pmatrix} m_{\nu_e} & m_{\nu_{e\mu}} \\ m_{\nu_{e\mu}} & m_{\nu_\mu} \end{pmatrix}$ .

The variation of the Lagrangian (56) under the  $SU(2)$  transformation:

$$\nu'_f(x) = e^{i\alpha_j \cdot \tau_j} \nu_f(x) \quad j = 1, 2, 3, \quad (57)$$

is given by

$$\delta \mathcal{L}_\nu(x) = i\alpha_j \bar{\nu}_f(x) [\tau_j, M_\nu] \nu_f(x) = -\alpha_j \partial_\mu J_{f,j}^\mu(x), \quad (58)$$

where

$$J_{f,j}^\mu(x) = \bar{\nu}_f(x) \gamma^\mu \tau_j \nu_f(x), \quad j = 1, 2, 3. \quad (59)$$

Again, the charges

$$Q_{f,j}(t) = \int d^3\mathbf{x} J_{f,j}^0(x) \quad (60)$$

close the  $su(2)$  algebra, however, because of the off-diagonal (mixing) terms in  $M_\nu$ ,  $Q_{f,3}(t)$  is time dependent. This implies an exchange of charge between  $\nu_e$  and  $\nu_\mu$ , resulting in the phenomenon of neutrino oscillations. The (time dependent) flavor charges for mixed fields are then defined as [19]:

$$Q_e(t) = \frac{1}{2}Q + Q_{f,3}(t), \quad Q_\mu(t) = \frac{1}{2}Q - Q_{f,3}(t), \quad (61)$$

$$Q_\sigma(t) = \int d^3\mathbf{x} \nu_\sigma^\dagger(x) \nu_\sigma(x), \quad (62)$$

where  $\sigma = e, \mu$  and  $Q_e(t) + Q_\mu(t) = Q$ .

### 3.1. Flavor states for mixed neutrinos

Till now our considerations have been essentially classical. We now quantize the fields with definite masses as usual (see Appendix), and consider the eigenstates of the above defined charges.

The normal ordered charge operators for free neutrinos  $\nu_1, \nu_2$  are:

$$:Q_i: \equiv \int d^3\mathbf{x} : \nu_i^\dagger(x) \nu_i(x) := \sum_r \int d^3\mathbf{k} \left( \alpha_{\mathbf{k},i}^{r\dagger} \alpha_{\mathbf{k},i}^r - \beta_{-\mathbf{k},i}^{r\dagger} \beta_{-\mathbf{k},i}^r \right), \quad (63)$$

where  $i = 1, 2$  and  $: \dots :$  denotes normal ordering with respect to the vacuum  $|0\rangle_{1,2}$ . The neutrino states with definite masses defined as

$$|\nu_{\mathbf{k},i}^r\rangle = \alpha_{\mathbf{k},i}^{r\dagger} |0\rangle_{1,2}, \quad i = 1, 2, \quad (64)$$

are clearly eigenstates of  $Q_1$  and  $Q_2$ , which can be identified with the lepton charges of neutrinos in the absence of mixing.



The situation is more delicate when mixing is present. In such a case, the flavor neutrino states have to be defined as the eigenstates of the flavor charges  $Q_\sigma(t)$  (at a given time). The relation between the flavor charges in the presence of mixing and those in the absence of mixing is:

$$Q_e(t) = \cos^2 \theta Q_1 + \sin^2 \theta Q_2 + \sin \theta \cos \theta \int d^3 \mathbf{x} \left[ \nu_1^\dagger(x) \nu_2(x) + \nu_2^\dagger(x) \nu_1(x) \right], \quad (65)$$

$$Q_\mu(t) = \sin^2 \theta Q_1 + \cos^2 \theta Q_2 - \sin \theta \cos \theta \int d^3 \mathbf{x} \left[ \nu_1^\dagger(x) \nu_2(x) + \nu_2^\dagger(x) \nu_1(x) \right]. \quad (66)$$

Notice that the last term in these expressions is proportional to the charge  $Q_{m,1}$  defined above (cf. Eq.(52)). The presence of such a term forbids the construction of eigenstates of the  $Q_\sigma(t)$  in the Hilbert space  $\mathcal{H}_{1,2}$ . One then is led [17] to define another Hilbert space,  $\mathcal{H}_{e,\mu}$  for the flavor neutrino fields. The flavor vacuum state  $|0\rangle_{e,\mu}$  and the mass vacuum state  $|0\rangle_{1,2}$  are orthogonal to each other (see Appendix).

The normal ordered flavor charge operators for mixed neutrinos are then written as

$$\begin{aligned} :: Q_\sigma(t) :: &\equiv \int d^3 \mathbf{x} :: \nu_\sigma^\dagger(x) \nu_\sigma(x) :: \\ &= \sum_r \int d^3 \mathbf{k} \left( \alpha_{\mathbf{k},\sigma}^{r\dagger}(t) \alpha_{\mathbf{k},\sigma}^r(t) - \beta_{-\mathbf{k},\sigma}^{r\dagger}(t) \beta_{-\mathbf{k},\sigma}^r(t) \right) \end{aligned} \quad (67)$$

where  $\sigma = e, \mu$ , and  $:: \dots ::$  denotes normal ordering with respect to  $|0\rangle_{e,\mu}$ . Thus, the flavor charges are diagonal in the flavor annihilation/creation operators constructed by means of the mixing generator  $G_\theta$  defined in the Appendix. The definition of the normal ordering  $:: \dots ::$  for any operator  $A$ , is the usual one:

$$:: A :: \equiv A - {}_{e,\mu} \langle 0 | A | 0 \rangle_{e,\mu}. \quad (68)$$

Note that  $:: Q_\sigma(t) :: = G_\theta^{-1}(t) : Q_j : G_\theta(t)$ , with  $(\sigma, j) = (e, 1), (\mu, 2)$ , and

$$:: Q :: = :: Q_e(t) :: + :: Q_\mu(t) :: = : Q_1 : + : Q_2 : = : Q :. \quad (69)$$

We define the flavor states as eigenstates of the flavor charges  $Q_\sigma$  at a reference time  $t = 0$ :

$$|\nu_{\mathbf{k},\sigma}^r\rangle \equiv \alpha_{\mathbf{k},\sigma}^{r\dagger}(0) |0(0)\rangle_{e,\mu}, \quad \sigma = e, \mu \quad (70)$$

and similar ones for antiparticles. We have

$$:: Q_e(0) :: |\nu_{\mathbf{k},e}^r\rangle = |\nu_{\mathbf{k},e}^r\rangle, \quad :: Q_\mu(0) :: |\nu_{\mathbf{k},\mu}^r\rangle = |\nu_{\mathbf{k},\mu}^r\rangle \quad (71)$$

$$:: Q_e(0) :: |\nu_{\mathbf{k},\mu}^r\rangle = 0 = :: Q_\mu(0) :: |\nu_{\mathbf{k},e}^r\rangle, \quad :: Q_\sigma(0) :: |0\rangle_{e,\mu} = 0. \quad (72)$$

The explicit form of the flavor states  $|\nu_{\mathbf{k},e}^r\rangle$  and  $|\nu_{\mathbf{k},\mu}^r\rangle$  at time  $t = 0$  is given in the Appendix.

### 3.2. Oscillation formulas in QFT

Flavor oscillation formulas are derived by computing, in the Heisenberg representation, the expectation value of the flavor charge operators on the flavor state. We have

$$\mathcal{Q}_{\nu_e \rightarrow \nu_\sigma}^{\mathbf{k}}(t) \equiv \langle \nu_{\mathbf{k},e}^r | Q_\sigma(t) | \nu_{\mathbf{k},e}^r \rangle = \left| \left\{ \alpha_{\mathbf{k},\sigma}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 + \left| \left\{ \beta_{-\mathbf{k},\sigma}^{r\dagger}(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|^2 \quad (73)$$

and

$${}_{e,\mu}\langle 0 | \ddot{Q}_e(t) | 0 \rangle_{e,\mu} = {}_{e,\mu}\langle 0 | Q_\mu(t) | 0 \rangle_{e,\mu} = 0, \quad (74)$$

The oscillation formulas are [18]:

$$\mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) = 1 - \sin^2(2\theta) \left[ |U_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right], \quad (75)$$

$$\mathcal{Q}_{\nu_e \rightarrow \nu_\mu}^{\mathbf{k}}(t) = \sin^2(2\theta) \left[ |U_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} - \omega_{k,1}}{2} t \right) + |V_{\mathbf{k}}|^2 \sin^2 \left( \frac{\omega_{k,2} + \omega_{k,1}}{2} t \right) \right]. \quad (76)$$

The charge conservation is ensured at any time:

$$\mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) + \mathcal{Q}_{\nu_e \rightarrow \nu_\mu}^{\mathbf{k}}(t) = 1. \quad (77)$$

The differences with respect to the Pontecorvo formulas are: the energy dependence of the amplitudes, and the additional oscillating term. In the relativistic limit:  $|\mathbf{k}| \gg \sqrt{m_1 m_2}$ , we have  $|U_{\mathbf{k}}|^2 \rightarrow 1$  and  $|V_{\mathbf{k}}|^2 \rightarrow 0$  and the traditional formulas are recovered.

#### 4. Entanglement in neutrino mixing – Quantum Field Theory

Following what done in the QM case, we now calculate the entanglement associated to an electron neutrino state at time  $t$ , by means of the variances of the above discussed charge operators.

Let us start with the Noether charges  $Q_{\nu_i}$ , which are expected to characterize the amount of static entanglement present in the states Eq.(70). We obtain:

$$\begin{aligned} \Delta Q_i(\nu_e)(t) &= \langle \nu_{\mathbf{k},e}^r | Q_i^2 | \nu_{\mathbf{k},e}^r \rangle - \langle \nu_{\mathbf{k},e}^r | Q_i | \nu_{\mathbf{k},e}^r \rangle^2 \\ &= \frac{1}{4} \sin^2(2\theta), \quad i = 1, 2. \end{aligned} \quad (78)$$

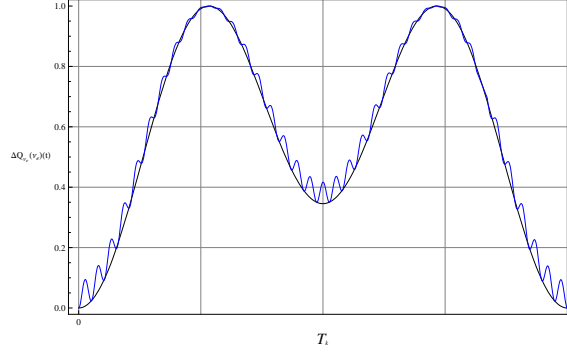
in perfect agreement with the quantum mechanical result Eq.(28).

Next we consider dynamic entanglement, which is described by the variances of the flavor charges. We have:

$$\begin{aligned} \Delta Q_e(\nu_e)(t) &= \langle \nu_{\mathbf{k},e}^r | Q_e^2(t) | \nu_{\mathbf{k},e}^r \rangle - \langle \nu_{\mathbf{k},e}^r | Q_e(t) | \nu_{\mathbf{k},e}^r \rangle^2 \\ &= \langle \nu_{\mathbf{k},e}^r | \left[ \sum_s \int d^3\mathbf{p} \left( \alpha_{\mathbf{p},e}^{s\dagger}(t) \alpha_{\mathbf{p},e}^s(t) - \beta_{-\mathbf{p},e}^{s\dagger}(t) \beta_{-\mathbf{p},e}^s(t) \right) \right]^2 | \nu_{\mathbf{k},e}^r \rangle - \left[ \mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) \right]^2 \\ &= \langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle + \langle \nu_{\mathbf{k},e}^r | \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle \\ &\quad - 2 \langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle - \left[ \mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) \right]^2. \end{aligned} \quad (79)$$

We now consider the third term in Eq.(79). We have:

$$\begin{aligned} \langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle &= {}_{e,\mu}\langle 0 | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu} \\ &\quad + \left| \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} \right|_{e,\mu}^2 \langle 0 | \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu} \\ &\quad - \left| \left\{ \alpha_{\mathbf{k},e}^r(0), \beta_{-\mathbf{k},e}^r(t) \right\} \right|_{e,\mu}^2 \langle 0 | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu} \\ &\quad - \left\{ \alpha_{\mathbf{k},e}^{r\dagger}(0), \beta_{-\mathbf{k},e}^{r\dagger}(t) \right\} \left\{ \alpha_{\mathbf{k},e}^r(0), \alpha_{\mathbf{k},e}^{r\dagger}(t) \right\} {}_{e,\mu}\langle 0 | \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^r(t) | 0 \rangle_{e,\mu} \\ &\quad + \left\{ \alpha_{\mathbf{k},e}^r(0), \beta_{-\mathbf{k},e}^r(t) \right\} \left\{ \alpha_{\mathbf{k},e}^r(t), \alpha_{\mathbf{k},e}^{r\dagger}(0) \right\} {}_{e,\mu}\langle 0 | \alpha_{\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) | 0 \rangle_{e,\mu} \end{aligned} \quad (80)$$



**Figure 2.** QM vs. QFT flavor entanglement for  $|\nu_e(t)\rangle$  as a function of the scaled time  $T = \frac{2Et}{\Delta m_{12}^2}$  with  $\theta$  fixed at the value  $\sin^2 \theta = 0.314$ .

Explicit calculation of the above quantity shows that:

$$\langle \nu_{\mathbf{k},e}^r | \alpha_{\mathbf{k},e}^{r\dagger}(t) \alpha_{\mathbf{k},e}^r(t) \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle = \langle \nu_{\mathbf{k},e}^r | \beta_{-\mathbf{k},e}^{r\dagger}(t) \beta_{-\mathbf{k},e}^r(t) | \nu_{\mathbf{k},e}^r \rangle \quad (81)$$

so that we have

$$\Delta Q_e(\nu_e)(t) = \mathcal{Q}_{\nu_e \rightarrow \nu_e}^{\mathbf{k}}(t) \mathcal{Q}_{\nu_e \rightarrow \nu_\mu}^{\mathbf{k}}(t) \quad (82)$$

which formally resembles the quantum mechanical result Eq.(44). The differences are now due to the presence of the flavor condensate, which affect the oscillation formulas (see Eqs.(75),(76)). In Fig.2, flavor entanglement formula is plotted in the QFT case against the corresponding QM case.

## 5. Conclusions

On the basis of recent results, we have discussed some aspect of entanglement in the phenomenon of neutrino mixing and oscillations. In the simple case of two flavor mixing, we have shown how to generalize previous results obtained in the context of quantum mechanics, to the case of quantum field theory. The difference between the QFT and the QM cases are related to the condensate vacuum structure associated to neutrino mixing in QFT.

Our study provides a simple, exactly solvable example for a possible extension of entanglement to quantum field theory. Apart from the differences with QM due to the flavor vacuum contributions, the QFT result is interesting from a more conceptual point of view. Indeed, both the static and the dynamical entanglement arise in connection with unitarily inequivalent representations: in the case of the static entanglement, the flavor Hilbert space at time  $t$  to which the entangled state  $|\nu_\sigma(t)\rangle$  belongs, is unitarily inequivalent to the Hilbert space for the qubit states  $|\nu_i\rangle$  [17]; on the other hand, in the case of dynamical entanglement, where the qubits are taken to be the flavor states at time  $t = 0$ , the inequivalence is among the flavor Hilbert space at different times [23]. In the first case, the relevant orthogonality relation is  $\lim_{V \rightarrow \infty} {}_m \langle 0|0(t) \rangle_f = 0$ , in the second  $\lim_{V \rightarrow \infty} {}_f \langle 0(t')|0(t) \rangle_f = 0$ , with  $t \neq t'$ .

Since the inequivalent representations are associated with a non-trivial condensate vacuum structure, the above conjecture suggests that, in the context of QFT, many interpretational issues connected with entanglement could be revisited in this new light.

## Appendix A. QFT formalism for mixed fields

The fields  $\nu_e(x)$  and  $\nu_\mu(x)$  are defined through the mixing relations (47), in terms of the free fields  $\nu_1(x)$  and  $\nu_2(x)$  which are expanded as

$$\nu_i(x) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} \left[ u_{\mathbf{k}, i}^r \alpha_{\mathbf{k}, i}^r(t) + v_{-\mathbf{k}, i}^r \beta_{-\mathbf{k}, i}^{r\dagger}(t) \right] e^{i\mathbf{k} \cdot \mathbf{x}}, \quad i = 1, 2 \quad (\text{A.1})$$

with  $\alpha_{\mathbf{k}, i}^r(t) = \alpha_{\mathbf{k}, i}^r e^{-i\omega_{\mathbf{k}, i} t}$ ,  $\beta_{\mathbf{k}, i}^{r\dagger}(t) = \beta_{\mathbf{k}, i}^{r\dagger} e^{i\omega_{\mathbf{k}, i} t}$ , and  $\omega_{\mathbf{k}, i} = \sqrt{\mathbf{k}^2 + m_i^2}$ . The operator  $\alpha_{\mathbf{k}, i}^r$  and  $\beta_{\mathbf{k}, i}^r$ ,  $i = 1, 2$ ,  $r = 1, 2$  are the annihilator operators for the vacuum state  $|0\rangle_m \equiv |0\rangle_1 \otimes |0\rangle_2$ :  $\alpha_{\mathbf{k}, i}^r |0\rangle_m = \beta_{\mathbf{k}, i}^r |0\rangle_m = 0$ . The anticommutation relations are:  $\{\nu_i^\alpha(x), \nu_j^{\beta\dagger}(y)\}_{t=t'} = \delta^3(\mathbf{x} - \mathbf{y}) \delta_{\alpha\beta} \delta_{ij}$ , with  $\alpha, \beta = 1, \dots, 4$ , and  $\{\alpha_{\mathbf{k}, i}^r, \alpha_{\mathbf{q}, j}^{s\dagger}\} = \delta_{\mathbf{k}\mathbf{q}} \delta_{rs} \delta_{ij}$ ;  $\{\beta_{\mathbf{k}, i}^r, \beta_{\mathbf{q}, j}^{s\dagger}\} = \delta_{\mathbf{k}\mathbf{q}} \delta_{rs} \delta_{ij}$ , with  $i, j = 1, 2$ . All other anticommutators vanish. The orthonormality and completeness relations are given by  $u_{\mathbf{k}, i}^{r\dagger} u_{\mathbf{k}, i}^s = v_{\mathbf{k}, i}^{r\dagger} v_{\mathbf{k}, i}^s = \delta_{rs}$ ,  $u_{\mathbf{k}, i}^{r\dagger} v_{-\mathbf{k}, i}^s = v_{-\mathbf{k}, i}^{r\dagger} u_{\mathbf{k}, i}^s = 0$ , and  $\sum_r (u_{\mathbf{k}, i}^r u_{\mathbf{k}, i}^{r\dagger} + v_{-\mathbf{k}, i}^r v_{-\mathbf{k}, i}^{r\dagger}) = \mathbb{I}$ .

The generator of the mixing transformations is given by [17]:

$$G_\theta(t) = \exp \left[ \theta \int d^3\mathbf{x} \left( \nu_1^\dagger(x) \nu_2(x) - \nu_2^\dagger(x) \nu_1(x) \right) \right] \quad (\text{A.2})$$

so that

$$\nu_\sigma^\alpha(x) = G_\theta^{-1}(t) \nu_i^\alpha(x) G_\theta(t); \quad (\sigma, i) = (e, 1), (\mu, 2) \quad (\text{A.3})$$

At finite volume, this is a unitary operator,  $G_\theta^{-1}(t) = G_{-\theta}(t) = G_\theta^\dagger(t)$ , preserving the canonical anticommutation relations. The generator  $G_\theta^{-1}(t)$  maps the Hilbert space for free fields  $\mathcal{H}_{1,2}$  to the Hilbert space for mixed fields  $\mathcal{H}_{e,\mu}$ :  $G_\theta^{-1}(t) : \mathcal{H}_{1,2} \mapsto \mathcal{H}_{e,\mu}$ . In particular, the flavor vacuum is given by  $|0(t)\rangle_{e,\mu} = G_\theta^{-1}(t) |0\rangle_{1,2}$  at finite volume  $V$ . We denote by  $|0\rangle_{e,\mu}$  the flavor vacuum at  $t = 0$ . In the infinite volume limit, the flavor and the mass vacua are unitarily inequivalent [17]. The explicit expression for  $|0\rangle_{e,\mu}$  at time  $t = 0$  in the reference frame for which  $\mathbf{k} = (0, 0, |\mathbf{k}|)$  is

$$\begin{aligned} |0\rangle_{e,\mu}^{\mathbf{k}} &= \prod_r \left[ (1 - \sin^2 \theta |V_{\mathbf{k}}|^2) - \epsilon^r \sin \theta \cos \theta |V_{\mathbf{k}}| (\alpha_{\mathbf{k}, 1}^{r\dagger} \beta_{-\mathbf{k}, 2}^{r\dagger} + \alpha_{\mathbf{k}, 2}^{r\dagger} \beta_{-\mathbf{k}, 1}^{r\dagger}) + \right. \\ &\quad \left. + \epsilon^r \sin^2 \theta |V_{\mathbf{k}}| |U_{\mathbf{k}}| (\alpha_{\mathbf{k}, 1}^{r\dagger} \beta_{-\mathbf{k}, 1}^{r\dagger} - \alpha_{\mathbf{k}, 2}^{r\dagger} \beta_{-\mathbf{k}, 2}^{r\dagger}) + \sin^2 \theta |V_{\mathbf{k}}|^2 \alpha_{\mathbf{k}, 1}^{r\dagger} \beta_{-\mathbf{k}, 2}^{r\dagger} \alpha_{\mathbf{k}, 2}^{r\dagger} \beta_{-\mathbf{k}, 1}^{r\dagger} \right] |0\rangle_{1,2} \end{aligned} \quad (\text{A.4})$$

The condensation density is given by

$${}_{e,\mu} \langle 0 | \alpha_{\mathbf{k}, i}^{r\dagger} \alpha_{\mathbf{k}, i}^r | 0 \rangle_{e,\mu} = {}_{e,\mu} \langle 0 | \beta_{\mathbf{k}, i}^{r\dagger} \beta_{\mathbf{k}, i}^r | 0 \rangle_{e,\mu} = \sin^2 \theta |V_{\mathbf{k}}|^2, \quad i = 1, 2. \quad (\text{A.5})$$

The flavor fields are written as:

$$\nu_\sigma(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}, r} e^{i\mathbf{k} \cdot \mathbf{x}} \left[ u_{\mathbf{k}, i}^r \alpha_{\mathbf{k}, \sigma}^r(t) + v_{-\mathbf{k}, i}^r \beta_{-\mathbf{k}, \sigma}^{r\dagger}(t) \right], \quad (\sigma, i) = (e, 1), (\mu, 2). \quad (\text{A.6})$$

The flavor annihilation operators are [17]:

$$\begin{aligned} \alpha_{\mathbf{k}, e}^r(t) &= \cos \theta \alpha_{\mathbf{k}, 1}^r(t) + \sin \theta \sum_s \left[ u_{\mathbf{k}, 1}^{r\dagger} u_{\mathbf{k}, 2}^s \alpha_{\mathbf{k}, 2}^s(t) + u_{\mathbf{k}, 1}^{r\dagger} v_{-\mathbf{k}, 2}^s \beta_{-\mathbf{k}, 2}^{s\dagger}(t) \right] \\ \alpha_{\mathbf{k}, \mu}^r(t) &= \cos \theta \alpha_{\mathbf{k}, 2}^r(t) - \sin \theta \sum_s \left[ u_{\mathbf{k}, 2}^{r\dagger} u_{\mathbf{k}, 1}^s \alpha_{\mathbf{k}, 1}^s(t) + u_{\mathbf{k}, 2}^{r\dagger} v_{-\mathbf{k}, 1}^s \beta_{-\mathbf{k}, 1}^{s\dagger}(t) \right] \\ \beta_{-\mathbf{k}, e}^r(t) &= \cos \theta \beta_{-\mathbf{k}, 1}^r(t) + \sin \theta \sum_s \left[ v_{-\mathbf{k}, 2}^{s\dagger} v_{-\mathbf{k}, 1}^r \beta_{-\mathbf{k}, 2}^s(t) + u_{\mathbf{k}, 2}^{s\dagger} v_{-\mathbf{k}, 1}^r \alpha_{\mathbf{k}, 2}^{s\dagger}(t) \right] \\ \beta_{-\mathbf{k}, \mu}^r(t) &= \cos \theta \beta_{-\mathbf{k}, 2}^r(t) - \sin \theta \sum_s \left[ v_{-\mathbf{k}, 1}^{s\dagger} v_{-\mathbf{k}, 2}^r \beta_{-\mathbf{k}, 1}^s(t) + u_{\mathbf{k}, 1}^{s\dagger} v_{-\mathbf{k}, 2}^r \alpha_{\mathbf{k}, 1}^{s\dagger}(t) \right]. \end{aligned} \quad (\text{A.7})$$

In the reference frame where  $\mathbf{k} = (0, 0, |\mathbf{k}|)$ , we have

$$\begin{aligned}\alpha_{\mathbf{k},e}^r(t) &= \cos \theta \alpha_{\mathbf{k},1}^r(t) + \sin \theta \left( |U_{\mathbf{k}}| \alpha_{\mathbf{k},2}^r(t) + \epsilon^r |V_{\mathbf{k}}| \beta_{-\mathbf{k},2}^{r\dagger}(t) \right), \\ \alpha_{\mathbf{k},\mu}^r(t) &= \cos \theta \alpha_{\mathbf{k},2}^r(t) - \sin \theta \left( |U_{\mathbf{k}}| \alpha_{\mathbf{k},1}^r(t) - \epsilon^r |V_{\mathbf{k}}| \beta_{-\mathbf{k},1}^{r\dagger}(t) \right), \\ \beta_{-\mathbf{k},e}^r(t) &= \cos \theta \beta_{-\mathbf{k},1}^r(t) + \sin \theta \left( |U_{\mathbf{k}}| \beta_{-\mathbf{k},2}^r(t) - \epsilon^r |V_{\mathbf{k}}| \alpha_{\mathbf{k},2}^{r\dagger}(t) \right), \\ \beta_{-\mathbf{k},\mu}^r(t) &= \cos \theta \beta_{-\mathbf{k},2}^r(t) - \sin \theta \left( |U_{\mathbf{k}}| \beta_{-\mathbf{k},1}^r(t) + \epsilon^r |V_{\mathbf{k}}| \alpha_{\mathbf{k},1}^{r\dagger}(t) \right).\end{aligned}\quad (\text{A.8})$$

Here  $\epsilon^r = (-1)^r$  and

$$\begin{aligned}|U_{\mathbf{k}}| &\equiv u_{\mathbf{k},i}^{r\dagger} u_{\mathbf{k},j}^r = v_{-\mathbf{k},i}^{r\dagger} v_{-\mathbf{k},j}^r = \frac{|\mathbf{k}|^2 + (\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}}, \quad i, j = 1, 2, \quad i \neq j, \\ |V_{\mathbf{k}}| &\equiv \epsilon^r u_{\mathbf{k},1}^{r\dagger} v_{-\mathbf{k},2}^r = -\epsilon^r u_{\mathbf{k},2}^{r\dagger} v_{-\mathbf{k},1}^r = \frac{(\omega_{k,1} + m_1) - (\omega_{k,2} + m_2)}{2\sqrt{\omega_{k,1}\omega_{k,2}(\omega_{k,1} + m_1)(\omega_{k,2} + m_2)}} |\mathbf{k}|,\end{aligned}\quad (\text{A.9})$$

with  $|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$ .

The explicit expressions for the flavor states  $|\nu_{\mathbf{k},e}^r\rangle$  and  $|\nu_{\mathbf{k},\mu}^r\rangle$  at time  $t = 0$ , in the reference frame for which  $\mathbf{k} = (0, 0, |\mathbf{k}|)$  are

$$\begin{aligned}|\nu_{\mathbf{k},e}^r\rangle &\equiv \alpha_{\mathbf{k},e}^{r\dagger}(0)|0\rangle_{e,\mu} \\ &= \left[ \cos \theta \alpha_{\mathbf{k},1}^{r\dagger} + |U_{\mathbf{k}}| \sin \theta \alpha_{\mathbf{k},2}^{r\dagger} - \epsilon^r |V_{\mathbf{k}}| \sin \theta \alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},1}^{r\dagger} \right] G_{\mathbf{k},s \neq r}^{-1}(\theta) \prod_{\mathbf{p} \neq \mathbf{k}} G_{\mathbf{p}}^{-1}(\theta) |0\rangle_{1,2},\end{aligned}\quad (\text{A.10})$$

$$\begin{aligned}|\nu_{\mathbf{k},\mu}^r\rangle &\equiv \alpha_{\mathbf{k},\mu}^{r\dagger}(0)|0\rangle_{e,\mu} \\ &= \left[ \cos \theta \alpha_{\mathbf{k},2}^{r\dagger} - |U_{\mathbf{k}}| \sin \theta \alpha_{\mathbf{k},1}^{r\dagger} + \epsilon^r |V_{\mathbf{k}}| \sin \theta \alpha_{\mathbf{k},1}^{r\dagger} \alpha_{\mathbf{k},2}^{r\dagger} \beta_{-\mathbf{k},2}^{r\dagger} \right] G_{\mathbf{k},s \neq r}^{-1}(\theta) \prod_{\mathbf{p} \neq \mathbf{k}} G_{\mathbf{p}}^{-1}(\theta) |0\rangle_{1,2},\end{aligned}\quad (\text{A.11})$$

where  $G(\theta, t) = \prod_{\mathbf{p}} \prod_{s=1}^2 G_{\mathbf{p},s}(\theta, t)$ . In these states a multiparticle component is present, disappearing in the relativistic limit  $|\mathbf{k}| \gg \sqrt{m_1 m_2}$ : in this limit, since  $|U_{\mathbf{k}}|^2 \rightarrow 1$  and  $|V_{\mathbf{k}}|^2 \rightarrow 0$ , the (quantum-mechanical) Pontecorvo states are recovered.

## Acknowledgments

We acknowledge partial financial support from MIUR, INFN, INFM and CNISM. M.B. thanks the organizers of the ‘‘Symmetries in Science Symposium - Bregenz 2009’’, for the very nice and creative atmosphere in which the workshop was held.

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